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Tidal Friction in the Irish Sea

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PHILOSOPHICAL TRANSACTIONS.

I. *Tidal Friction in the Irish Sea.*

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The dissipation of energy in the tides has recently formed the subject of a paper by Mr. R. O. Street.* In that paper it is assumed that the energy is dissipated by the viscous drag of layers of water which move parallel to the bottom of the sea. The assumption that tidal currents move in laminar motion is so opposed to ordinary observation of the surface of the sea in a tideway that I felt certain, on reading the paper, that if some other method could be found, which did not depend on any special assumptions as to the nature of the motion, it would be found that Mr. Street's estimate of the dissipation is very much too small.

This view is strengthened by the consideration that Reynolds' criterion† of stability would lead us to expect that eddies would form in any stream of sea-water flowing at a speed of 1 knot or more, when the depth is greater than some quantity of the order of magnitude of 1 or 2 cm. Since the mean depth of the Irish Sea is over 40 fathoms, mathematical considerations alone would lead us to suspect the existence of the eddies, which can in fact be seen marking the surface of the sea in places where the current runs exceptionally strongly, or over a particularly uneven bottom. Several of these places are marked as "ripples" on the chart of the Irish Sea, the sheet of water to which Mr. Street applied his calculations.

Dissipation of Energy in Tidal Currents.

The mechanism by means of which energy is dissipated in a tidal current by friction on the bottom must be similar to the mechanism by which the energy of a river is dissipated by friction on its bed, and also to the mechanism by which the energy of the wind is dissipated by friction on the ground. The amount of friction in both these cases is known. It can in both cases be expressed by a term of the form F, the skin-friction per square centimetre, which is equal to $K\rho V^2$, where ρ is

* 'Roy. Soc. Proc.,' A, vol. 93, 1917, p. 349.

† See Osborne Reynolds, "On the Dynamic Theory of Incompressible Viscous Fluid and the Determination of the Criterion," 'Phil. Trans.,' A, 1894, p. 123.

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the density of the fluid, V its velocity, and K is a constant depending on the nature of the surface.

Friction on the Bed of a River.—A very large amount of work has been done on the friction of a river on its bed. The results of these researches have been used to make various empirical formulæ. One of the best known of these is that of BAZIN, which takes the following form

$$rs = \frac{1}{7569}\left(1-\frac{\gamma}{\sqrt{r}}\right)^2 V^2, \quad \ldots \quad (1)$$

where r is the "hydraulic radius" of the channel, *i.e.*, the area divided by the wetted part of the perimeter of the cross-section, s is the slope of the bed, γ is a constant which depends on the nature of the bottom. In this engineering formula metres are used instead of centimetres as the unit of length. In order to find out the relationship between this formula and one of the type

$$F = K\rho V^2 \quad \ldots \quad (2)$$

one must equate the resistance acting up-stream to the component of the weight of the fluid acting down-stream. This gives

$$F \times (\text{wetted part of perimeter}) = s\rho g \times (\text{area of cross-section}),$$

or

$$\frac{K\rho V^2}{\rho g} = rs. \quad \ldots \quad (3)$$

Comparing this with (1) it will be seen that

$$K = \frac{g}{7569}\left(1-\frac{\gamma}{\sqrt{r}}\right)^2.$$

But $g = 9{\cdot}81$ expressed in metre-second units.

Hence

$$K = 0{\cdot}0013\left(1+\frac{\gamma}{\sqrt{r}}\right)^2,$$

where K is non-dimensional. In the case of the Irish Sea, to which this formula will be applied, the depth is about 80 metres. In the case of a stream which is very broad compared with its depth, the depth and the hydraulic radius are the same thing. Hence for the Irish Sea $\sqrt{r} = \sqrt{80} = 9$, approximately.

The value of γ depends on the nature of the bottom. For a clean stony, or smooth earth bottom, BAZIN* gives $\gamma = 0{\cdot}85$. Taking this value as being applicable to the Irish Sea,

$$K = 0{\cdot}0013\left(1+\frac{0{\cdot}85}{9}\right)^2 = 0{\cdot}0016. \quad \ldots \quad (4)$$

* See 'Cours d'Hydraulique,' J. GRIALOU, Paris, 1916.

In places where the bottom is uneven or weedy, BAZIN gives 1·7 as the value of γ. Under these circumstances

$$K = 0{\cdot}0013\left(1+\frac{1{\cdot}7}{9}\right)^2 = 0{\cdot}0018. \qquad (5)$$

It will be seen that large changes in the amount of roughness produce only small changes in the amount of friction on the bottom. On looking at BAZIN's formula it will be seen that this is due to the fact that the sea is deep. In order that the roughness of the bottom may have a large effect in slowing down a stream, it is necessary that r should be small. It seems, in fact, that the size of the projections which constitute the roughness or inequality of the bed must be some definite fraction of r in order that their effect may be felt on the stream as a whole. In other words, the direct effect of the projections extends to a distance which is some multiple of the linear dimension of the projections; and if these are small enough compared with the depth, very little difference is made to the total flow of the stream by changing the amount of roughness on the bottom.

This conclusion is important in the present application, because it means that by adopting the values of K given above we shall be able to get a fairly accurate estimate of the friction of the sea on the bottom without knowing the exact nature of the bottom. We may under-estimate the friction, but we are certainly not likely to over-estimate it; for our estimate will not take account of unevennesses, such as boulders and rocks, which are comparable with the depth of the sea, nor will it take account of the increase in K in the shallow areas of estuaries and outlying banks.

Friction of the Wind on the Ground.—It has already been pointed out that the friction of the sea on the sea-bottom is similar to the friction of the wind on the ground. According to the principle of dynamical similarity the flow-patterns of the sea and air will be the same, provided the scale of the projections which constitute the roughness are the same, and provided

$$\frac{v_w}{v_a} = \frac{\mu_w \rho_a}{\mu_a \rho_w}, \qquad (6)$$

where ρ_a and ρ_w are the densities of air and sea-water respectively, μ_a and μ_w are their viscosities, and v_a and v_w are their velocities. Using values obtained from physical tables $\mu_w\rho_a\,(\mu_a\rho_w)$ will be found to be equal to $\frac{1}{11}$.

In a previous communication to the Royal Society* the author has shown from meteorological observations that the friction of the wind over the grass land of Salisbury Plain may be expressed by means of the formula $F = 0{\cdot}002\rho_a v_a^2$ over the whole range of velocities tested, *i.e.*, from 6 to 30 miles per hour.

According to the principle of dynamical similarity therefore this same expression may be expected to apply to tidal currents of $\frac{6}{11}$ to $\frac{30}{11}$ miles per hour, *i.e.*, roughly

* 'Roy. Soc. Proc.,' A, vol. 92, p. 196, 1916.

$\frac{1}{2}$ to 3 knots. This is the very range of speed with which we have to deal in tidal measurements. Hence, if we assume that the roughness of the bottom of the sea is about the same as that of the grass land of Salisbury Plain, the formula

$$F = 0{\cdot}002\rho v^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for the friction of a tidal stream, of velocity v, on the sea-bottom may be expected to give reasonably accurate results.

It will be noticed that the value of K, 0·002, is very nearly the same as the values 0·0016 and 0·0018 obtained from experiments and observations on the flow of large rivers. It also agrees fairly well with laboratory experiments on the friction of air and water in pipes and with experiments on the friction of flat surfaces in water.

Calculation of the Energy Dissipated by Tidal Friction.—We can now proceed to calculate the amount of energy dissipated by tidal currents in the Irish Sea, the sheet of water which it is proposed to discuss.

The rate of dissipation of energy by friction is equal to the friction multiplied by the relative velocity of the surfaces between which the friction acts. Using the expression $F = K\rho v^2$ for the friction of the current on the bottom, the amount of energy dissipated per square centimetre per second is therefore

$$K\rho v^3.$$

The currents in the Irish Sea vary from place to place, and also with the varying state of the tide. It is necessary therefore to find the average value of $K\rho v^3$ during a tidal period, and then to take the average value of this expression over the whole area considered.

The tidal stream at any time t, after it has attained its maximum velocity, may be taken roughly as $v = V \cos \frac{2\pi t}{T}$, where V is the maximum tidal stream and T is the semi-diurnal tidal period of 12h. 25m.

The average rate of dissipation of energy over each square centimetre of the Irish Sea is therefore equal to the mean value of

$$K\rho V^3 \cos^3 \frac{2\pi t}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The average value of $\cos^3 \frac{2\pi t}{T}$ taken without regard to sign is $4/3\pi$.

The average value of V^3 over the Irish Sea could be obtained from tidal measurements. Mr. STREET, in the paper already referred to, has found the average value of V^2 at spring tides over the Irish Sea. His estimate is 5 (knots)^2. This would make $V = 2\frac{1}{4}$ knots. If we assume this as the value of V in (8) we shall not be far from the truth, because the variability of the maximum streams in the Irish Sea is not sufficiently great to give rise to much difference between the square root of the mean square of the velocity and the mean velocity, or between this and the cube root of the mean cube of the velocity. By taking $V = 2\frac{1}{4}$ knots = 114 cm. per second in (8) we slightly under-estimate the friction ; we shall not in any case over-estimate it.

Using in (8) the value $K = 0{\cdot}002$,* and remembering that ρ, the density of sea-water, is 1·03, it will be found that w, the mean rate of dissipation of energy, per square centimetre per second, in the Irish Sea at spring tides is

$$w = 0{\cdot}002\,(1{\cdot}03)\,(114)^3\left(\frac{4}{3\pi}\right) = 1300 \text{ ergs per square centimetre per second.} \qquad (9)$$

Using the least admissible value of K† it will be found that

$$w = 1040 \text{ ergs per square centimetre per second.} \quad . \quad . \quad . \quad . \quad (10)$$

Mr. STREET's estimate, when reduced to C.G.S. units, is 7 ergs per square centimetre per second, which is only $\frac{1}{150}$th of our minimum estimate.‡

Rate at which Energy enters the Irish Sea owing to the Action of External Forces.

The amount of dissipation found by the method just described is so different from that obtained by Mr. STREET, and so much larger than any previous estimate of tidal friction that I have come across, that it seemed worth while to try and verify it, if possible, by some different method. Instead of trying to measure the rate of dissipation at every point of the Irish Sea, I have calculated the rate at which energy enters the Irish Sea through the North and South Channels. To this must be added the rate at which work is done by lunar attraction on the waters of the Irish Sea. The sum of these will give the rate at which the energy of that sea is increasing, *plus* the rate of dissipation of energy. When the average values of these expressions are taken during a complete tidal period it is evident that, since the energy of the Irish Sea does not increase or decrease continually, the average value of the rate of increase in energy is zero. Hence the average rate of dissipation of energy by the tidal currents can be found.

Rate at which Energy flows into the Irish Sea.—The calculation of the rate at which energy flows into the Irish Sea is very simple. Consider the flow of energy across the surface, S (fig. 1), formed by the vertical lines between a closed curve, s, on the surface of the sea and its projection on the bottom.

* See equation (7).

† See equation (4).

‡ Mr. STREET informs me that since publishing the paper already referred to, he has obtained other results which confirm his previous results. He hopes to publish them when circumstances permit.

Let ds be an element of length of the curve, s,

v = velocity of current at any point on s,

θ = angle between element ds and direction of current,

D = depth of bottom below mean sea-level,

h = height of tide above mean sea-level,

ρ = density of sea-water,

g = acceleration due to gravity.

First consider the rate at which energy is communicated to the portion of the sea which, at time, t, was enclosed by the surface, S. Let S′ be the moving surface which encloses this water.

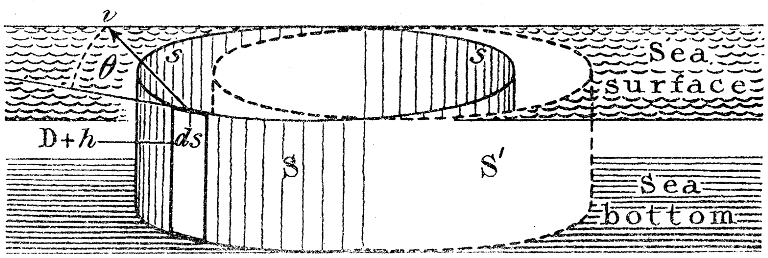

Fig. 1. Diagram showing successive positions of a surface, S, formed by the vertical lines through the curve, s, as that surface moves with the current.

The mean hydrostatic pressure on a vertical strip of S, of height D+h and width ds is $\rho g \frac{1}{2}(\mathrm{D}+h)$. Its area is $(\mathrm{D}+h)\,ds$. The work done by hydrostatic pressure on the portion of the sea originally enclosed in the surface is therefore

$$\int \rho g\left(\frac{\mathrm{D}+h}{2}\right)(v \sin\theta\, dt)\,(\mathrm{D}+h)\, ds. \qquad (11)$$

This then is the amount of energy which has flowed during the time, dt, through the surface, S′, which originally coincided with the fixed surface, S, but which moves with the fluid.

To find the amount of energy which has crossed the original surface, S, during the time, dt, it is necessary to take account of the energy contained in the fluid which has actually crossed the fixed surface, S. The gravitational potential energy of a vertical column of fluid, of height D+h and horizontal cross-section $v \sin\theta\, dt\, ds$, is evidently $\rho g\,(\mathrm{D}+h)\left(\frac{h-\mathrm{D}}{2}\right) v \sin\theta\, dt\, ds$, mean sea-level being regarded as the surface of zero potential. The kinetic energy of the same column is

$$\tfrac{1}{2}\rho v^2\,(\mathrm{D}+h)\, v \sin\theta\, dt\, ds.$$

The amount of energy in the fluid which crosses the element of surface $d\mathrm{S}$ during the time dt is the sum of these two. The amount in the fluid which crosses the whole surface, S, is therefore

$$\int \tfrac{1}{2}\rho v \sin\theta \, dt \,\{gh^2 - g\mathrm{D}^2 + v^2(\mathrm{D}+h)\}\, ds, \quad \ldots \quad (12)$$

where the integral is taken round the curve, s.* The amount of energy which crosses the surface, S, in time dt is the sum of (11) and (12), that is,

$$\int \tfrac{1}{2}\rho v \sin\theta \, dt \,\{g(\mathrm{D}+h)^2 + gh^2 - g\mathrm{D}^2 + v^2(\mathrm{D}+h)\}\, ds$$

$$= \rho g\, dt \int \mathrm{D}hv \sin\theta \, ds + \int \tfrac{1}{2}\rho v \sin\theta \, dt\,(2gh^2 + \mathrm{D}v^2 + hv^2)\, ds. \quad \ldots \quad (13)$$

We shall now assume that h is small compared with D. This is true for the Irish Sea where the average maximum rise of tide above the mean sea-level is about 6 feet (one fathom) at spring tides, while the average depth is over 40 fathoms.

It is evident also that since the order of magnitude of v must be that of ch/D, where c is the velocity of a tidal wave in water of depth D (*i.e.*, $c = \sqrt{g\mathrm{D}}$),

* It has been suggested to me that a term should be added to allow for the potential energy of the entering water due to the moon's attraction. This appears to be a misapprehension. Potential energy is only a mathematical expression used in finding the work done on matter by certain systems of forces.

The work done in time δt by the moon's attraction on the liquid contained in any surface which is fixed relatively to the earth is

$$-\delta t \iiint \rho \frac{d\Omega}{dt}\, dv, \quad \ldots \quad (\mathrm{A})$$

where Ω is the potential due to the moon's attraction, dv is an element of volume, and the integration extends throughout the volume. If the linear dimensions of the surface are small, so that Ω does not vary appreciably throughout its volume (this may be taken as true for the Irish Sea), then $\iiint \rho\, dv = \mathrm{M}$, the mass of the liquid contained in the surface at any time.

The total work done by the moon in a complete period is $-\int \mathrm{M}\frac{d\Omega}{dt}\,dt = -\int \mathrm{M}\,d\Omega$.

Integrating by parts $-\int \mathrm{M}\,d\Omega = -[\mathrm{M}\Omega] + \int \Omega\, d\mathrm{M}$, where $[\mathrm{M}\Omega]$ represents the change in the product $\mathrm{M}\Omega$ during a complete period. This is evidently equal to 0. Hence

$$-\int \mathrm{M}\,d\Omega = \int \Omega\, d\mathrm{M} = \int \Omega \frac{d\mathrm{M}}{dt}\, dt, \quad \ldots \quad (\mathrm{B})$$

$\frac{d\mathrm{M}}{dt}$ is the rate at which water enters the volume and $\Omega\frac{d\mathrm{M}}{dt}$ is the potential energy of the entering water.

In calculating the work done by the moon on the waters of the Irish Sea we could therefore use either expression A or expression B, but we must not use *both*.

At a later stage in this paper (see p. 18) the work done by the moon's attraction has been calculated from expression A. The potential energy, due to the moon's attraction, of the entering water has therefore been left out at the present stage.

therefore all the terms in the second integral of (13) are small compared with those of the first.

We shall therefore neglect

$$\int \tfrac{1}{2}\rho v \sin\theta\, dt\, (2gh^2 + \mathrm{D}v^2 + hv^2)\, ds$$

in comparison with

$$\rho g\, dt \int \mathrm{D}hv \sin\theta\, ds. \qquad (14)$$

Taking account of the conservation of energy, this must be equal to the sum of the increase in kinetic energy of the sea included in the area enclosed by s, the energy dissipated during the time dt by tidal friction, and the work done by the moon's attraction during the same time. It has already been pointed out that since there is no continual increase in the kinetic energy of any portion of the sea, the first of these will vanish when we come to consider the mean rate of dissipation of energy over a complete tidal period.

If W is the average rate at which energy is dissipated by tidal friction in the portion of the sea enclosed by s, and W_m is the average rate at which work is done on it by the moon's attraction, it will be seen from (14) that

$$\mathrm{W} - \mathrm{W}_m = \text{average value of } \left\{ g\rho \int \mathrm{D}hv \sin\theta\, ds \right\}. \qquad (15)$$

Application to the Irish Sea.

In applying this expression to the Irish Sea, it will be necessary to evaluate the integrals across sections of the North and South Channels; and in choosing the exact positions of these sections, it is clear that those parts of the channels must be selected where the greatest number of observations of the rise of tide and the strengths of the currents have been made.

The only observations of tidal currents in the Irish Sea to which I have had access are contained in the Admiralty publication 'Tides and Tidal Streams of the British Islands.'* The observations on the rise and fall of tide are contained in the 'Admiralty Tide Tables' and the 'Irish Coast Pilot.'

Height of the Tide.—The Tide Tables give the time of H.W. at full and change of the moon. They also give the range of tide at spring tides and at neaps. They afford no indication, as a rule, of the height of the tide at the intermediate hours, except when there is some marked peculiarity such as the long-continued H.W. at Poole, or the bore in the Bristol Channel. The principal tidal phenomenon is, however, the semi-diurnal rise and fall of tide, with a period of 12h. 25m., and in a

* First edition, 1909.

large majority of cases this can be represented with sufficient accuracy for most purposes by a term of the form h, the height of the tide above mean sea-level

$$= \mathrm{H} \cos \frac{2\pi}{\mathrm{T}} (t + \mathrm{T}_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where

2H is the range of tide between H.W. and L.W.

T is the tidal period of 12h. 25m.

t is the time measured from the time of the moon's passage over the Greenwich meridian. At full and change of the moon, t is Greenwich mean time.

T_1 is the time of H.W. at full and change of the moon, *i.e.*, the "establishment" of the place in question. We shall henceforth assume that h can be expressed by means of the equation (16).

In evaluating the integral (15) it will be seen that it is necessary to know the height of the tide at all points on the section. Unfortunately nearly all the measurements of rise and fall of tide have been made on the coast. None have been made in the middle of the channel, or at any rate none are recorded in the tables.

At first sight we might expect tidal range to be the same on the two sides of a channel, but this not the case. On the opposite sides of the South Channel, at the entrance to the Irish Sea, for instance, the tidal ranges at spring tides are 4 feet at Arklow on the Irish side and 15 feet at Bardsey Island on the Welsh side. This is not an accidental circumstance connected with particular formations of the coast in the neighbourhood of Bardsey or Arklow; all the tidal ranges in the neighbourhood show the same characteristic. On the Irish coast there is Arklow with a tidal range of 4 feet; Courtown, $3\frac{3}{4}$ feet; Arklow Bank, $4\frac{1}{4}$ feet; and Kilmichael Point, $4\frac{3}{4}$ feet; while on the Welsh coast there are St. Tudwall Road, 14 feet; Port Dynllayn, $12\frac{1}{4}$ feet; Llanddwyn Island, $14\frac{1}{2}$ feet; Bardsey Island, 15 feet; and Holyhead, 16 feet. In evaluating the integral (15), therefore, it is important to know how the tidal range varies from the Welsh to the Irish coasts. In other words, does the level change more rapidly near the Welsh or near the Irish coast, or does the sea at H.W. slope uniformly down, and at L.W. slope uniformly up, from Bardsey to Arklow? In deciding this question, dynamical considerations are of great assistance.

The reason for the difference in the range on the two sides of a channel is well known; it is connected with the "geostrophic" force, due to the earth's rotation, which tends to deflect bodies moving on the earth's surface to the right in the Northern, and to the left in the Southern Hemisphere. The flood stream into the Irish Sea cannot be deflected to the right because of the Welsh coast. The water therefore piles up on that side till the hydrostatic pressure-gradient is sufficient to keep the water moving straight. The same reasoning applies to the ebb stream which piles itself up against the Irish coast. At the particular section from Arklow to Bardsey the flood stream is a maximum at H.W. and the ebb stream a maximum at L.W. Hence the effect of the slope of the sea surface, which is necessary to keep the stream straight against the deflecting force due to the earth's rotation, is to add to H.W. and to subtract from L.W. on the Welsh side, thus increasing the range above the mean range for the section. The effect on the Irish side is exactly the reverse, so that the tidal range is diminished there. Though this explanation is given in general terms it is a simple matter to express the forces and slopes concerned in a quantitative manner.

The application to the present question follows directly. If it can be shown by observation that the tidal currents move straight up and down the channel without being deflected across it, then the slope of the sea surface must everywhere correspond with the velocity of the current. If the current is nearly uniform right across the channel, then the sea will slope down uniformly from one side of the channel to the other. It will be shown later, in discussing the tidal currents, that both these conditions are satisfied. Dynamical considerations therefore enable us to say what the tidal range in mid-channel is, when we know it at either side.

Confidence in the correctness of this view is greatly strengthened by calculating the difference to be expected in the tidal ranges on the two sides of the channel, and showing that it is in close agreement with the observed difference.

The deflecting force due to the earth's rotation which acts on each cubic centimetre of the sea is $2\omega\rho v \sin\lambda$, where ω is the angular velocity of the earth's rotation, and λ is the latitude.

The slope of the surface, in a direction perpendicular to the stream, which will just balance this force, is therefore

$$\frac{2\omega\rho v \sin\lambda}{\rho g}, \quad \text{or} \quad \frac{2\omega v \sin\lambda}{g} \quad \ldots\ldots\ldots \quad (17)$$

The measured maximum velocity of both the flood and the ebb stream at spring tides across the section, AB, from Arklow to Bardsey* is 3·2 knots,† = 162 cm. per second; $\omega = 0·000073$; in latitude 52°, $\sin\lambda = 0·79$, $g = 981$. Hence from (17) the slope is $1·9 \times 10^{-5}$ radians.

The distance across the channel in a direction perpendicular to the current from Bardsey Island to Arklow, on the Irish coast, is 48 nautical miles = 288,000 feet. Hence the difference in level at time of the maximum current between the sea surface at Bardsey Island and at Arklow should be $1·9 \times 10^{-5} \times 2·88 \times 10^{5} = 5·7$ feet.

Now, as has been mentioned already, the streams in this part of the Irish Sea have their maximum velocities at H.W. and L.W. The curves shown in fig. 2 represent

* See map, fig. 3.

† See p. 12 later.

the velocities of the tidal streams at various states of the tide at various lightships in the neighbourhood of the Arklow-Bardsey section.

On inspecting the curves shown in that figure, it will be seen that the maximum current is at about 3h. before H.W. at Dover. It is H.W. at Dover at 11h. 7m. full and change, and it is H.W. on the Arklow-Bardsey line at 8h. 10m.* full and change. Hence it is H.W. on AB 3h. before H.W. at Dover, *i.e.*, at the time of the maximum tidal current.

Since H.W. coincides with the time of maximum current, the difference in range at spring tides between those at Arklow and those at Bardsey should be $2 \times 5 \cdot 7$ feet $= 11 \cdot 4$ feet. The measured range at Bardsey at spring tides is 15 feet, while that at

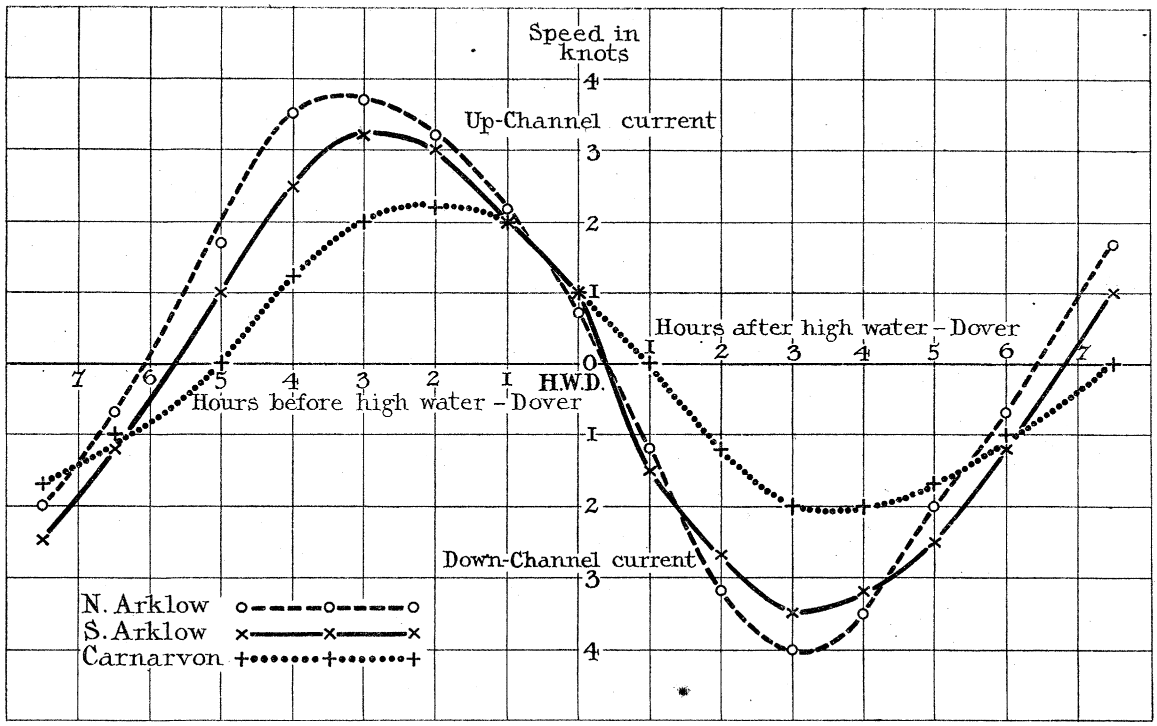

Fig. 2. Curves showing the velocity of the tidal currents at three light ships in the South Channel to the Irish Sea at various states of the tide.

Arklow is 4 feet. The difference, 11 feet, is almost exactly equal to the calculated difference $11\frac{1}{2}$ feet.

The accuracy with which this calculation is verified by observation is very good evidence that the sea actually slopes in the way we should expect from the current measurements, that is to say, uniformly from Bardsey Island to Arklow. It is not necessary, therefore, for our purpose to have actual tidal measurements in mid-channel, though it is to be hoped that these conclusions will some day be tested by observation.

* See p. 15 later.

The considerations just advanced show that if a channel is so narrow that the water is forced to travel straight up and down it, then the difference in level between the water on the two sides may be calculated on the assumption that the sea slopes to an extent which gives rise to a pressure gradient across the channel which is exactly equal to the deflecting force due to the earth's rotation. If the channel is rather wider the central parts of the stream may be able to move across the channel slightly. This would reduce the slope. In the case of the South Channel of the Irish Sea, however, these cross currents are very small, as may be seen by examining the figures given in the table on p. 14, where it is shown that the direction of the current is practically constant during the ebb and during the flood streams. We are therefore justified in assuming that the South Channel is narrow enough to allow us to apply the calculations given above.

Velocity of the Tidal Currents.—We now come to the measurements of tidal currents. These are the principal factors which determine our choice of sections suitable for measuring the flow of energy into the Irish Sea.

South Channel.—In the South Channel the best section is that shown as AB in the map (fig. 3). It runs from Bardsey Island through the south end of Arklow Bank. Along this section tidal measurements have been made at the points marked in the map as S_1, S_2, S_3, S_4.

In the position S_1, 5 miles from Arklow Bank, the maximum velocities of the ebb and flood streams are both 3·6 knots. The direction of the flood stream is N. 32° E., while that of the ebb is S. 26° W.

At S_2, 15 miles from Arklow Bank, the maximum flood stream is N. 35° E. at 3·2 knots, while the maximum ebb stream is S. 32° W. at 3·3 knots.

At S_3, 15 miles from Bardsey Island, the maximum flood stream is N. 25° E., 3·2 knots, while the maximum ebb is S. 28° W., 3·0 knots.

At S_4, 5 miles from Bardsey Island, the maximum flood stream is N. 16° E., 3·0 knots, while the maximum ebb is S. 16° W., 2·3 knots.

It will be seen, therefore, that the maximum current velocity is nearly constant along the section, its average value being 3·2 knots. The direction also varies very little; the average direction of the flood stream being N. 27° E., while that of the ebb S. 26° W. These are practically opposite directions. They will (for simplicity) be assumed to be exactly opposite during the rest of this discussion.

No measurements of the speed and direction of the currents at the points S_1, S_2, S_3, S_4, are given for the intermediate hours; but several such measurements are given for other points in the neighbourhood. I have selected three of these sets which were taken at the nearest points to the section AB. They were made at the South Arklow, North Arklow and Carnarvon Bay light-ships, respectively.

In the accompanying table* are given the velocities and directions of the current

* Taken from 'Tides and Tidal Streams of the British Islands,' first edition, 1909.

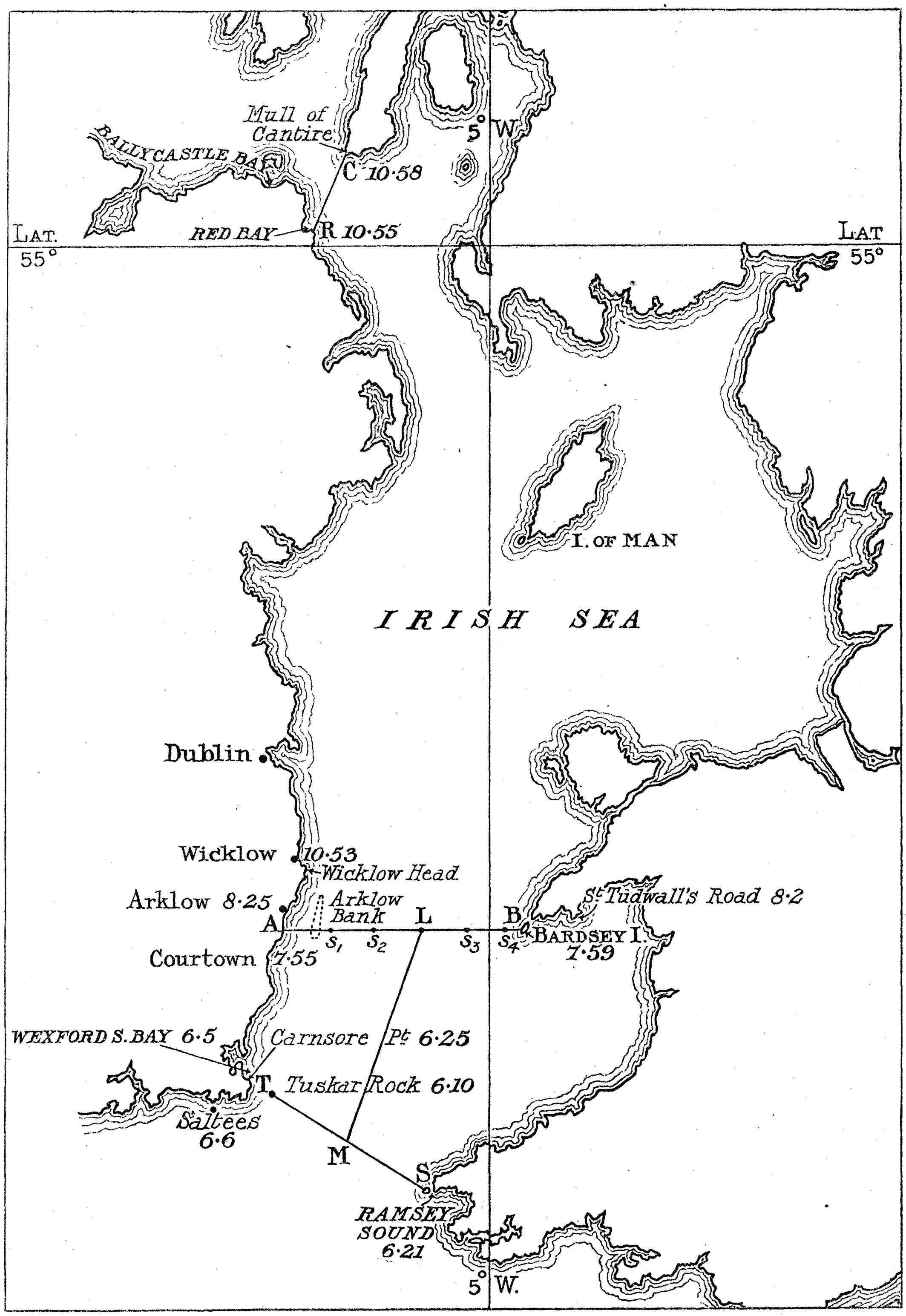

Fig. 3. Map of the Irish Sea. The figures are the times of H.W. at full and change of the moon. Thus Courtown 7·55 means that at Courtown it is H.W. at 7h. 55m. at full and change of the moon.

at various hours referred to H.W., Dover. It will be seen that in all three sets of measurements the current remains constant in direction during the flood and during the ebb streams, except for a short period, just about the time of H.W., in the case of the South Arklow measurements when there is a change in direction of about 10°. The direction of the flood stream is exactly opposite to that of the ebb stream in this region.

TABLE showing the Direction and Velocity of the Tidal Streams at Three Light-Ships at Various States of the Tide.

		North Arklow light vessel.		South Arklow light vessel.		Carnarvon Bay light vessel.	
		Diurnal magnitude.	Rate. Knots.	Diurnal magnitude.	Rate. Knots.	Diurnal magnitude.	Rate. Knots.
Hours before H.W. at Dover	5	N. 43° E.	1·7	N. 43° E.	1·0	Slack	—
	4	N. 43° E.	3·5	N. 43° E.	2·5	N. 21° E.	1·2
	3	N. 43° E.	3·7	N. 43° E.	3·2	N. 21° E.	2·0
	2	N. 43° E.	3·2	N. 43° E.	3·0	N. 21° E.	2·2
	1	N. 43° E.	2·2	N. 43° E.	2·0	N. 21° E.	2·0
H.W. at Dover		N. 43° E.	0·7	N. 54° E.	1·0	N. 21° E.	1·0
Hours after H.W. at Dover	1	S. 43° W.	1·2	S. 32° W.	1·5	Slack	—
	2	S. 43° W.	3·2	S. 43° W.	2·7	S. 21° W.	1·2
	3	S. 43° W.	4·0	S. 43° W.	3·5	S. 21° W.	2·0
	4	S. 43° W.	3·5	S. 43° W.	3·2	S. 21° W.	2·0
	5	S. 43° W.	2·0	S 43° W.	2·5	S. 21° W.	1·7
	6	S. 43° W	0·7	S. 43° W.	1·2	S. 21° W.	1·0

The variation of current velocity with the state of the tide is shown in the curves in fig. 2, which are drawn from the measurements recorded in the above table. On inspecting the curves of fig. 2 it will be seen that in the neighbourhood of the section AB the tidal streams can be represented sufficiently accurately for many purposes by a sine curve. We can, therefore, express the current by the mathematical expression

$$v = \mathrm{V} \cos \frac{2\pi}{\mathrm{T}} (t + \mathrm{T}_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where V is the maximum current velocity, which in the case of the section AB is 3·2 knots, t has the same meaning as before (see p. 9), and T_0 is the time of maximum current at full and change of the moon. In the case of the section AB this must be 2h. 45m. before H.W., Dover, since the stream changes direction at 15m. after H.W., Dover. Since H.W., Dover, at full and change of moon is at 11h. 7m., therefore, T_0 = 8h. 20m. approximately.

Height of Tide at Section AB.—Having now chosen the section AB of the South Channel along which we intend to calculate the average value of $\left\{g\rho\int \mathrm{D}hv \sin \theta ds\right\}$ we must return to the discussion of the values of h. In the first place the line AB is practically a co-tidal line, *i.e.*, a line through all points at which it is H.W. simultaneously. On the Irish side it is H.W. at Arklow Bank at 8h. 24m.; at Arklow at 8h. 25m.; at Kilmichael Point, where the section AB strikes the Irish coast, at 8h. 25m.; and at Courtown, about 4 miles south of Kilmichael Point, at 7h. 55m. On the Welsh side it is H.W. at St. Tudwall Road at 8h. 2m., at Bardsey Island at 7h. 59m. The time of H.W. all along the line AB may therefore be taken as 8h. 10m., which is the mean of the times at either end. This will only be a few minutes wrong at either end, and the convenience, in evaluating the integral, of assuming a constant time of H.W. along the section, is very great. The value of T_1 will therefore be taken as 8h. 10m. all along AB.

In the expression $h = \mathrm{H} \cos \frac{2\pi}{\mathrm{T}}(t+\mathrm{T}_1)$ which was adopted to give the height of the tide at any point, the value of H varies from point to point.

It has been shown, however, that H must decrease uniformly from one side of the channel to the other, and that in the case of the section AB, where there is practically no difference between the time of H.W. and the time of the maximum flood stream, the sea at H.W. slopes at an angle $\frac{2\omega v \sin \lambda}{g}$.

If y is the distance of any point from the central line of the channel measured in the direction perpendicular to the current and towards the Irish Coast, then

$$\mathrm{H} = \mathrm{H}_1 - \frac{2\omega v \sin \lambda}{g} y, \qquad (19)$$

where H_1 is half the range of tide in mid-channel. We have already seen that the ranges of tide on the two sides of the channel are 4 and 15 feet; hence $\mathrm{H}_1 = \frac{1}{2}\left(\frac{15+4}{2}\right) = 4\frac{3}{4}$ feet. If s is the distance measured from the central point, L, of AB, then

$$s \sin \theta = y. \qquad (20)$$

Evaluation of the Rate of Transfer of Energy through the South Channel of the Irish Sea.

We are now in a position to evaluate W_{ab}, the average rate at which energy enters the Irish Sea across the section AB,

$$\mathrm{W}_{ab} = \text{average value of} \left\{g\rho \int_{\mathrm{A}}^{\mathrm{B}} \mathrm{D}hv \sin \theta \, ds\right\}. \qquad (21)$$

Substituting in (21) from (18), (19) and (20),

$$W_{ab} = \text{average value of} \left\{ g\rho \int_A^B D \left(H_1 - \frac{2\omega v \sin\lambda}{g} s \sin\theta \right) \cos\frac{2\pi}{T}(t+T_1)\, V \cos\frac{2\pi}{T}(t+T_0) \sin\theta\, ds \right\}.$$

Since the only terms which contain t are

$$\cos\frac{2\pi}{T}(t+T_1) \quad \text{and} \quad \cos\frac{2\pi}{T}(t+T_0),$$

we can integrate them with respect to t to find the average value of the main integral. Thus the average value of

$$\cos\frac{2\pi}{T}(t+T_1)\cos\frac{2\pi}{T}(t+T_0)$$

is evidently

$$\tfrac{1}{2}\cos\frac{2\pi}{T}(T_1-T_0).$$

Hence taking out all the quantities which are nearly constant across the section

$$W_{ab} = \tfrac{1}{2} g\rho V \sin\theta \cos\frac{2\pi}{T}(T_1-T_0) \int_A^B D \left(H_1 - \frac{2\omega v \sin\lambda}{g} s \sin\theta \right) ds. \quad . \quad . \quad (22)$$

To evaluate this it is only necessary to measure the depths at all points across the section. Actually this is not really necessary, for the depth is nearly uniform across the section AB, the average depth being 37 fathoms.

Under these circumstances, since the origin of s is taken at mid-channel, the value of $\int_A^B D \frac{2\omega v \sin\lambda}{g} s \sin\theta\, ds$ is zero. If the channel had not happened to be nearly uniform in depth, it would have been possible to evaluate this integral from the charted depths across the section. Hence

$$W_{ab} = \tfrac{1}{2} g\rho V \sin\theta \cos\frac{2\pi}{T}(T_1-T_0)\, DH_1L, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

where L is the length of AB, 50 nautical miles. The numerical values of the other constituents which occur in (23) are

$g = 981$.

ρ = density of sea water = 1·03.

V = 3·2 knots = 163 cm. per second.

θ = angle between current and direction of AB.

AB runs in a direction N. 86° E., while the current runs in a direction N. 26° E., so that $\theta = 60°$ and $\sin\theta = 0{\cdot}87$.

$$T = 12{\cdot}4\text{h.}, \qquad T_1 = 8\text{h. } 10\text{m.}, \qquad T_0 = 8\text{h. } 20\text{m.}$$

so that

$$T_1 - T_0 = 10\text{m.} = \tfrac{1}{6}\text{th h.},$$

and

$$\cos\frac{2\pi}{T}(T_1 - T_0) = \cos\left(\frac{180}{6\times 12{\cdot}4}\right)^0 = \cos 2{\cdot}4° = 1{\cdot}0,$$

$$D = 37 \text{ fathoms} = 6800 \text{ cm.}$$

$$H_1 = 4\tfrac{3}{4} \text{ feet} = 145 \text{ cm.}$$

$$L = 50 \text{ nautical miles} = 9{\cdot}1\times 10^6 \text{ cm.}$$

Hence the mean rate at which energy is transmitted across the section AB is

$$W_{ab} = \tfrac{1}{2}\times 981\times 1{\cdot}03\times 163\times 0{\cdot}87\times 1{\cdot}0\times 6800\times 145\times 9{\cdot}1\times 10^6$$
$$= 6{\cdot}4\times 10^{17} \text{ ergs per second.} \qquad (24)$$

North Channel.—The same method may be applied to the North Channel, but it is at once obvious that practically no energy enters the Irish Sea through this channel. The tidal streams set strongly through the North Channel, running in from 5h. to 11h. and out from 11h. to 5h. at full and change of the moon. The neck between the Mull of Cantyre and the Irish Coast forms a loop in a stationary oscillation. It is H.W. at Mull of Cantyre at 10h. 58m. At Red Bay, on the Irish Coast, it is H.W. at 10h. 55m. The co-tidal line for 10h. 55m., therefore, runs from the Mull of Cantyre to Red Bay, and it is for this reason that it has been chosen for the section RC (see fig. 3) along which the integral for W_{RC} (the energy which flows across RC) will be taken. Since the streams change direction at H.W., Dover, *i.e.*, at 11h. 7m., the phase difference between the tidal stream and the height of the tide is only 12m. of time short of the quarter period, *i.e.*, 87° expressed as an angle.

The maximum current runs through the North Channel at a rate of 4 knots. The rise and fall of tide in the North Channel is very small; at Red Bay it is 4 feet, and at Ballycastle Bay, to the N.W. of Red Bay, it is only 3 feet. At the Mull of Cantyre it is also 4 feet. The equality of the heights of the tide on the two sides of the channel is probably due to the fact that, at the times the stream is running at its maximum speed, when therefore we should expect the maximum difference in level on the two sides of the Channel owing to geostrophic force, the water is at its mean level. At H.W. and L.W. the streams are slack, so that no geostrophic effect is to be anticipated at those times.

In the formula

$$W_{RC} = \rho g \frac{DVH}{2} \cos\theta \cos\frac{2\pi}{T}(T_0 - T_1)\times(\text{length of RC}) \qquad (25)$$

for the rate of flow of energy into the Irish Sea, across the section RC, the numerical values of the terms are

$$H = \tfrac{1}{2}\,(4 \text{ feet}) = 61 \text{ cm.}$$

$$\frac{2\pi}{T}(T_0 - T_1) = 87^\circ, \text{ so that } \cos \frac{2\pi}{T}(T_0 - T_1) = 0{\cdot}05$$

$$V = 4 \text{ knots} = 200 \text{ cm. per second.}$$

(Length of RC) $\times \cos\theta$ is evidently equal to the breadth of the North Channel normal to the stream. This is 11 nautical miles, or 2×10^6 cm.

D_1 the mean depth, is about 65 fathoms $= 10^4$ cm. Hence the mean rate at which energy enters the Irish Sea by the North Channel is

$$W_{RC} = \tfrac{1}{2} \times 1{\cdot}03 \times 981 \times 10^4 \times 200 \times 61 \times 2 \times 10^6 \times 0{\cdot}05 = 6{\cdot}2 \times 10^{15} \text{ ergs per second.}$$

This is only $\frac{1}{100}$th of the energy which enters by the South Channel. It is obvious that no high degree of accuracy is aimed at in obtaining this figure. It is merely intended to show that the amount of energy which enters the Irish Sea by the North Channel is quite insignificant compared with the amount which enters by the South Channel. In the work which follows, I shall neglect it altogether, and shall consider merely the South Channel.

Amount of Work Done by the Moon's Attraction on the Waters of the Irish Sea.

The attraction of the moon may be expressed by means of a potential function Ω. Consider the work done by the moon's attraction on the water contained in an element of volume, Δ, which is fixed to the earth's surface. If the element contains water during two complete tidal periods, *i.e.* till it comes back to its original position relative to the moon, no work will be done on it. If on the other hand, the element, Δ, is situated within the space which is filled with water at high-tide and is empty at low-tide, work may be done on the water contained in Δ.

If ρ be the density of sea-water, h the height of the tide above mean sea-level, the work done by the moon's attraction during two complete lunar semi-diurnal tides, on a column of sea of 1 sq. cm. cross-section and stretching from the sea bottom to the surface is evidently

$$m = \int h\rho \, d\Omega, \qquad (26)$$

the integral extending over all the changes in Ω which occur during the complete cycle. Evidently the total energy communicated by the moon's attraction during two periods is

$$E_M = \iint m \, d\sigma, \qquad (27)$$

$d\sigma$ being an element of surface, and the integral extending over the whole surface of the Irish Sea included between the two sections AB and RC.

The potential of the moon's attraction on the sea is represented by the function

$$\Omega = \tfrac{3}{2}\daleth \frac{MR^2}{D_m^3}\left(\tfrac{1}{3} - \cos^2 \vartheta\right)^* \quad \ldots \quad (28)$$

where

$\daleth$ is the constant of gravitation.
M is the mass of the moon.
D_m is the radius of the moon's orbit.
R is the earth's radius.
ϑ is the angle between the line joining the centre of the earth to the moon, and the radius of the earth which passes through the point on the earth's surface which is being considered.

If λ be the latitude of the place (*i.e.*, 52° in the case of the Irish Sea), and if ϕ be the angle through which the earth has turned relative to the radius vector to the moon, since the moon was on the meridian, then by spherical trigonometry,

$$\cos \vartheta = \cos \lambda \cos \phi. \quad \ldots \quad (29)$$

Also, if 2H be the range of tide at the place which is being considered,

$$h = H \cos 2(\phi + \phi_0), \quad \ldots \quad (30)$$

where ϕ_0 is the phase of the tide at the time when the moon crosses the meridian.

Combining (26), (28), (29), (30), it will be seen that

$$m = -\rho H \tfrac{3}{2}\daleth \frac{MR^2}{D_m^3}\cos^2\lambda \int_0^{2\pi} \cos 2(\phi+\phi_0)\, d(\cos^2\phi)$$

$$= -\tfrac{3}{2}\pi\rho H \daleth \frac{MR^2}{D_m^3}\sin^2\phi_0 (\cos^2\lambda). \quad \ldots \quad (31)$$

(31) may be written

$$m = -\tfrac{3}{2}\pi\rho H \cos^2\lambda \sin^2\phi_0 \left(\frac{\daleth E}{R^2}\right)\left(\frac{M}{E}\right)\left(\frac{R^3}{D_m^3}\right) R,$$

where E is the mass of the earth.

$\frac{\daleth E}{R^2}$ is the attraction of the earth at its surface, *i.e.*,

$$\frac{\daleth E}{R^2} = g = 981 \text{ in C.G.S. units.}$$

* See LAMB'S 'Hydrodynamics,' p. 339 (1906 edition).

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$\frac{M}{E}$, the ratio of the masses of the moon and the earth, is $\frac{1}{81}$; $\frac{R}{D_m}$, the ratio of the radius of the earth to the radius of the moon's orbit, is $\frac{1}{60}$.

$$\cos^2\lambda = 0{\cdot}38,$$
$$\rho = 1{\cdot}03,$$
$$R = 6{\cdot}4 \times 10^8 \text{ cm.}$$

Hence

$$m = -\tfrac{3}{2} \times \pi \times 1{\cdot}03 \times 0{\cdot}38 \times 981 \times \tfrac{1}{81} \times (\tfrac{1}{60})^3 \times 6{\cdot}4 \times 10^8 \times H \sin^2\phi_0$$
$$= -6{\cdot}6 \times 10^4 H \sin^2\phi_0 \text{ ergs.} \qquad (32)$$

The mean rate at which energy is communicated by lunar attraction is found by dividing this by $8{\cdot}7 \times 10^4$, the number of seconds in two semi-diurnal periods, *i.e.*, in 24h. 50m.

Hence w_M, the mean rate at which work is done by the moon's attraction on each square centimetre of the Irish Sea, is

$$w_M = \frac{-6{\cdot}6 \times 10^4}{8{\cdot}7 \times 10^4} \times (\text{average value of } H \sin^2\phi_0 \text{ over the Irish Sea}).$$

Now the mean value of 2H, the rise and fall of tide in the Irish Sea, is about 14 feet or 420 cm. Hence H may be taken as 210 cm. The average time of H.W. is about $1\frac{1}{2}$h. before the moon's meridian passage. Hence $\phi_0 = +22\frac{1}{2}^\circ$ and $\sin^2\phi_0 = 0{\cdot}7$.

A rough approximation to the average value of $H \sin^2\phi_0$ is therefore $210 \times 0{\cdot}7 = 150$ cm.; hence

$$w_M = \frac{-6{\cdot}6}{8{\cdot}7} \times 150 = -110 \text{ ergs per square centimetre per second.} \qquad (33)$$

It will be noticed that since it is H.W. shortly before the moon's meridian passage, work is done by the tides in the Irish Sea on the moon. This is indicated by the negative sign in (33).

Dissipation of Energy in the Irish Sea.

We have now seen that the rate at which energy flows into the Irish Sea through the North and South Channels is $6{\cdot}4 \times 10^{17}$ ergs per second.

The area of the Irish Sea between the sections AB and RC is 11,600 square nautical miles $= 3{\cdot}9 \times 10^{14}$ sq. cm. Hence energy enters the Irish Sea through the Channels at a rate of $\frac{6{\cdot}4 \times 10^{17}}{3{\cdot}9 \times 10^{14}} = 1640$ ergs per second per square centimetre of its area. Of this energy we have just seen (see equation 33) that 110 ergs per square centimetre per second are used in doing work against the moon's attraction. The remainder $1640-110 = 1530$ ergs per second are dissipated by tidal friction.

It will be remembered that the estimates previously given from a direct consideration of skin friction were 1040* and 1300† ergs per square centimetre per second.

It will be seen that the agreement between the two methods of estimating the energy dissipation due to tidal friction is quite remarkable. The conclusion appears inevitable that the dissipation of energy by tidal friction is very much larger than has previously been supposed.

Proportion of the Tidal Wave which is Absorbed in the Irish Sea.

The large amount of tidal energy which these calculations show to be absorbed in the Irish Sea naturally gives rise to speculations as to how large a proportion of the energy of the tidal wave is absorbed, and how much of it is reflected back to the Atlantic.

It has generally been believed that tidal friction plays only a very small part in tidal phenomena. Reasoning on the lines of the present work, however, the mere fact that it is possible to find a section where the rise and fall of tide is appreciable and where the phases of the tidal current and tidal height are the same, is a proof that an appreciable proportion of the energy of the tidal wave entering through the South Channel is absorbed. We cannot measure the size of the tidal wave which enters the Irish Sea directly, because at all points the effects of the entering and of the emerging wave will be felt simultaneously. It is necessary therefore to disentangle those effects. When this has been done it will be found that apparently complex tidal phenomena in the South Channel are really very simple.

It has often been pointed out that the Irish Sea behaves like a resonator with two open ends. The rise and fall of tide at the two open ends, which are "loops" in the oscillation, is small, while the current is a maximum at these points. In the middle of the Irish Sea in the neighbourhood of the Isle of Man, the currents are small while the rise and fall of tide is large. If the motion of the sea at the two channels is at all similar to a "loop" in a stationary oscillation, it may evidently be analysed into two waves, one going in and the other coming out.

We shall assume that the motion of the water in the South Channel is all backwards and forwards along the channel, as in fact the current observations show it to be. We have seen that the effect of the deflecting force due to the earth's rotation is to increase the tide on one side of the channel and to decrease it on the other, leaving the rise-and-fall of the central part of the channel unaffected. We shall deal first with the tidal phenomena which do not depend on this deflecting force, by considering the motion in the central part of the channel.

* See equation (10).

† See equation (9).

Let us assume that the tidal phenomena in the South Channel can be represented by the superposition of two waves, one of amplitude a going in, and the other of amplitude b, going out. These may be represented mathematically by the formula

$$h = a\cos\frac{2\pi}{\mathrm{T}}\left(t-\frac{x}{c}\right)-b\cos\frac{2\pi}{\mathrm{T}}\left(t+\frac{x}{c}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

where the first term represents the wave entering the channel, and the second represents the reflected wave leaving the channel. $2a$ and $2b$ are the ranges of the tides due to the two waves separately; x is the distance measured along the channel in the direction of the Irish Sea; and c is the velocity of a long wave in water of the depth, D, of the channel so that $c = \sqrt{g\mathrm{D}}$.*

Our problem is to analyse the observed tidal phenomena so as to find the values of a and b, and to show that the various characteristic features of the tidal phenomena of the South Channel can be accounted for by considering them as being due to these two waves.

The current due to the entering tidal wave is $a\sqrt{\frac{g}{\mathrm{D}}}\cos\frac{2\pi}{\mathrm{T}}\left(t-\frac{x}{c}\right)$.† The current due to the out-going tidal wave is $b\sqrt{\frac{g}{\mathrm{D}}}\cos\frac{2\pi}{\mathrm{T}}\left(t+\frac{x}{c}\right)$.

They are both positive at $x = 0$ if a and b are both positive, because the original formula (34) assumed for h, gives h as the difference of two terms and not as the sum.

Hence the tidal current, v, is

$$v = a\sqrt{\frac{g}{\mathrm{D}}}\cos\frac{2\pi}{\mathrm{T}}\left(t-\frac{x}{c}\right)+b\sqrt{\frac{g}{\mathrm{D}}}\cos\frac{2\pi}{\mathrm{T}}\left(t+\frac{x}{c}\right).$$

* It has been suggested that the velocity of the waves into and out of the Irish Sea are not equal to $\sqrt{g\mathrm{D}}$, because they are forced waves due to the moon. The moon's attraction, however, does not appear to be capable of exerting sufficient force to alter appreciably the velocity of a free wave of the amplitude with which we are concerned travelling down a channel of a depth of about 37 fathoms.

If f be the horizontal component of the moon's attraction, the maximum possible value of f is $8 \cdot 57 \times 10^{-8} g$ (see LAMB'S 'Hydrodynamics,' 4th edition, p. 256).

The maximum value of the horizontal force F, due to the pressure gradient in a free wave of height $2a$ from trough to crest, is $\mathrm{F} = \frac{2\pi}{\sqrt{g\mathrm{D}}\mathrm{T}}$, where T is the tidal period of 12·4h.

It will be seen later that the semi-amplitude of the smaller of the two waves with which we are concerned, *i.e.*, the out-going wave, is 145 cm. Taking $a = 145$, D = 37 fathoms = 6800 cm., it will be found that $\mathrm{F} = 8 \times 10^{-6} g$.

It appears therefore that f, the horizontal force due to the moon's attraction, is only $\frac{1}{100}$th of F, the force due to the horizontal pressure gradient in a free wave of the height with which we are concerned. The velocity of these waves cannot therefore differ appreciably from that of free waves in the channel.

† This is the well-known formula connecting the current velocity and tidal range in a tidal wave.

The tidal current is a maximum when $x = 0$ and $t = 0$. Its value is then

$$V = (a+b)\sqrt{\frac{g}{D}} \quad \ldots \ldots \ldots \quad (35)$$

At the point $x = 0$ the phases of the current and of the height of the tide are the same. In applying these equations to the channels of the Irish Sea, we must choose as origin, $x = 0$, the place where the height of the tide and the current have the same phase. As we have already seen in the South Channel, this point must be very close to the section AB from Arklow Bank to Bardsey Island, for the phase difference in that region is only 10m. of time. It is hardly worth while in the present investigation to take account of so small a difference as 10m.

At the point $x = 0$, the range of tide, which we have called $2H_1$, is evidently $2(a-b)$. Now we know the values of V, g and D; we can therefore calculate $a+b = V\sqrt{\frac{D}{g}}$ from (35), a and b can therefore be found separately.

Using the values already given for the section AB,

$$H_1 = \tfrac{1}{2} \text{ tidal range } = 4\tfrac{3}{4} \text{ feet } = 145 \text{ cm.}$$

$$D = 37 \text{ fathoms} = 6800 \text{ cm.}$$

$$g = 981.$$

$$V = 3{\cdot}2 \text{ knots} = 163 \text{ cm. per second,}$$

it will be seen that

$$a+b = 163\sqrt{\frac{6800}{981}} = 430 \text{ cm.} \quad \ldots \ldots \ldots \quad (36)$$

and

$$a-b = H_1 = 145 \text{ cm.} \quad \ldots \ldots \ldots \quad (37)$$

Hence, solving (36) and (37) we get for the semi-amplitudes of the in- and out-going tidal waves,

$$\left.\begin{aligned} a &= 287 \text{ cm.} \\ b &= 143 \text{ cm.} \end{aligned}\right\}$$

and

$$\frac{a}{b} = 2{\cdot}0. \quad \ldots \ldots \ldots \quad (38)$$

It appears, therefore, that at spring tides, the tidal wave is reduced almost exactly to half its amplitude during its passage into and out of the Irish Sea. The wave which comes out of the Irish Sea therefore contains only quarter of the energy of the wave which goes in.

The result just obtained does not appear to be open to any theoretical objection, but it is opposed to the generally accepted view that tidal friction has very little effect on the regime of the tides. It is worth while, therefore, to try and confirm it in some other way. With this end in view, we shall discuss the movement of the co-tidal lines in the South Channel.

First, let us consider the theoretical aspects of the case. A co-tidal line is a line at all points of which it is H.W. at the same time. In a progressive wave the co-tidal lines are the positions of the crest of the wave at a series of successive times. The distance apart of the co-tidal lines corresponding with a series of times, separated by intervals of 1h., will be a measure of the velocity of the wave. In drawing a map of co-tidal lines, therefore, one is apt to think that they represent the successive stages of advancement of a progressive tidal wave. This idea is incorrect. In the case of two superposed waves moving in opposite directions, for instance, it will be found that the co-tidal line moves in the same direction as that one of the two waves which has the greater amplitude; but that it does not move at the same speed.

In certain places the line moves faster than the wave, while in others it moves more slowly, and a knowledge of the relationship between the velocity of the co-tidal line and the velocity of the wave will enable us to determine the ratio of the amplitudes of the two waves.

The height of the tide above mean sea-level at any time is

$$h = a \cos\frac{2\pi}{\mathrm{T}}\left(t-\frac{x}{c}\right)-b \cos \frac{2\pi}{\mathrm{T}}\left(t+\frac{x}{c}\right).$$

This may be written in the form

$$h = \mathrm{A} \cos \frac{2\pi}{\mathrm{T}}(t-t_x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

where

$$\mathrm{A} = \sqrt{a^2+b^2-2ab \cos \frac{4\pi x}{c\mathrm{T}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

and

$$\cot \frac{2\pi t_x}{\mathrm{T}} = \frac{a-b}{a+b} \cot \frac{2\pi x}{c\mathrm{T}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

The co-tidal line, therefore, moves in time t_x, through the distance x, from the place where the phases of current and tide are the same, x and t_x are related by the equation (41).

The velocity, V_c, of the co-tidal line is therefore obtained by differentiating (41)

$$\mathrm{V}_c = \frac{dx}{dt_x} = c\,\frac{\left(\frac{a-b}{a+b}\right)^2 \cot^2 \frac{2\pi x}{c\mathrm{T}} + 1}{\cot^2 \frac{2\pi x}{c\mathrm{T}} + 1}\left(\frac{a+b}{a-b}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

At the point $x = 0$ where the tidal heights of the two waves oppose, and the tidal streams concur, the velocity of the co-tidal line is therefore a fraction $\frac{a-b}{a+b}$ of the velocity of the tidal wave. On examining the data at our disposal it will be found that they are hardly sufficient to place the co-tidal lines for 6, 7, 8, 9 and 10 hours sufficiently accurately on the map to make an accurate determination of the velocity of the co-tidal line in the neighbourhood of the line AB.

On looking at maps of co-tidal lines for the Irish Sea, however, such as that given in KRUMMEL'S 'Ozeanographie,'* it will be seen that the co-tidal lines for successive hours are crowded together in the neighbourhood of the Arklow-Bardsey line. V_c is therefore a minimum in that region, as we should expect from (42).

Though we cannot measure accurately the velocity of the co-tidal line as it passes the Arklow-Bardsey section, there are two sections of the channel where the positions of single co-tidal lines can be determined with considerable accuracy. We can therefore determine the mean velocity of the co-tidal line between these two sections and can compare this with theory. The line AB, which is practically a co-tidal line for 8h. 10m. is one example. Bardsey Island at its eastern end is separated from the mainland, so that the error due to a shelving shore is lessened. The time of H.W. at Courtown, a few miles south of the point where the western end of AB strikes the Irish coast, is 8h., while the time of H.W. at Arklow Bank and Arklow, both a little north of AB, and therefore a little later in their tides, is 8h. 25m. Greenwich mean time.

The co-tidal line for 8h. 10m. is therefore well determined and is practically coincident with AB. As before we shall take it as being coincident with $x = 0$.

The other co-tidal line referred to is the one for 6h. 15m. It is H.W. at Carnsore Point at 6h. 25m., and at Tuskar Rock, $4\frac{3}{4}$ miles off the Irish coast, at 6h. 10m., while the other end of the line is determined by Ramsey Sound, off the Welsh coast, where it is H.W. at 6h. 21m. The co-tidal line for 6h. 15m. is shown as the line TS in the map, fig. (3).

Let us then apply the formula (41) to find the ratio of the amplitudes of the two tidal waves. The distance between the mid points, M and L of the two co-tidal lines AB and TS, is about 43 nautical miles, so that in (41) $x = -43$ miles. The mean depth of the water between the two sections is 45 fathoms, and the velocity of a long wave in water of this depth is 56 nautical miles per hour. Remembering that T, the period of the semi-diurnal tide is 12h. 25m. or 12·4h., it will be found that

$$\cot\frac{2\pi x}{c\mathrm{T}} = \cot\frac{2\pi\,(-43)}{56\times 12{\cdot}4} = \cot\,(-22{\cdot}3^\circ) = -2{\cdot}44. \quad . \quad . \quad . \quad . \quad (43)$$

Also $t_x = 6\text{h. }15\text{m.} - 8\text{h. }10\text{m.} = -1{\cdot}92\text{h.}$ and

$$\cot\frac{2\pi t_x}{\mathrm{T}} = \cot\frac{2\pi\,(-1{\cdot}92)}{12{\cdot}4} = \cot\,(-56^\circ) = -0{\cdot}67. \quad . \quad . \quad . \quad . \quad (44)$$

* 'Handbuch der Ozeanographie,' vol. 2, p. 336 (1911 edition).

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Hence from (41)

$$\frac{a-b}{a+b}=\frac{\cot\dfrac{2x\pi}{c\mathrm{T}}}{\cot\dfrac{2\pi t_x}{\mathrm{T}}}=\frac{2\cdot44}{0\cdot67},$$

so that

$$\frac{a}{b}=\frac{2\cdot44+0\cdot67}{2\cdot44-0\cdot67}=1\cdot8. \qquad (45)$$

The agreement between this and the previous result $\frac{a}{b}=2\cdot0$,* is remarkable, because they are based on quite different data.

At this point it is worth while to look back at what we have done. We began by assuming that the tides in the South Channel of the Irish Sea can be represented by two tidal waves moving in opposite directions and with the velocity appropriate to the depth of the channel, *i.e.* $\sqrt{g\mathrm{D}}$.

We then used two totally different methods for finding the ratio of the rise and fall of tide due to each of the two waves.

The first of these methods depends on the relationship between the tidal currents, the depth and the rise and fall of tide across the section where the tidal current and tidal height are in the same phase.

The tidal currents have been measured at four points across the section in question. The depths of course are well known and are marked on all charts. The height of the range of the tide has only been measured at points at, and near the ends of, the section. The fact that the tidal current moves backwards and forwards in a straight line and that it is practically uniform across the section, is however a strong reason for believing that the tidal range decreases uniformly from the Welsh to the Irish shore. It is worth pointing out, however, that the mean rise and fall of tide across the section is not a measured quantity; it is a deduction, based on dynamical conceptions it is true, but still a deduction, from the measured amounts of the rise and fall of tide at each end of the section, and the measured tidal currents across it. The ratio of the amplitudes of the in- and out-going tidal waves was found by this method to be 2·0.

The second method of determining the ratio of the amplitude of the in-going tidal wave to the out-going wave depends on the ratio of the rate of movement of the co-tidal line to the velocity of the tidal wave. It was possible to get two well determined positions of the co-tidal line, one at each end of the South Channel. From measurements of the distance between the mid points of these two lines, and the interval of time between the two H.W's., the ratio of the amplitudes of the two waves in mid channel was found to be 1·8, almost exactly the same result as that obtained by the other method. It is worth noticing that this second method

* See equation (38).

involves only measurements of depth and time of H.W. It does not involve measurements of current or tidal range at all.

The remarkable agreement between these two methods of estimating the loss of energy in the tidal wave during its passage into and out of the Irish Sea is strong evidence that three-quarters of the energy of the tidal wave entering by the South Channel is dissipated in the Irish Sea. The main purpose of this paper is therefore accomplished.

To complete the investigation, however, it is worth while to show that certain other tidal phenomena of the South Channel, hitherto apparently not very well understood, are simple consequences of the superposition of two tidal waves of different amplitudes moving in opposite directions.

The first of these is the difference between the velocities of the co-tidal line on the two sides of the channel. This difference is very marked. The distance from Tuskar Rock to the south end of Arklow Bank, traversed by the co-tidal line up the Irish Coast during the interval between 6h. 15m. and 8h. 10m. is only 30 nautical miles, while the distance from Ramsey Island to Bardsey Island traversed by the co-tidal line in the same interval on the Welsh Coast is 59 miles, so that the velocity of the co-tidal line on one side of the channel is about double its velocity on the other. This difference evidently causes the co-tidal line to turn through a large angle, independently of any turning which the wave fronts themselves may experience in passing through the channel owing to a difference between the depths on the two sides.

In the case of the South Channel the depths are practically equal on the two sides, so the fronts of the waves will not turn, though of course they may be very slightly convex, owing to the greater depth in mid-channel.

The direction of the line AB from South Arklow to Bardsey is N. 86° W. The direction of the line TS from Tuskar to Ramsey is S. $57\frac{1}{2}$° W. The angle turned through by the co-tidal line from 6h. 15m. to 8h. 10m. is therefore N. 86° W. *minus* S. 57° W. = $36\frac{1}{2}$°.

As a matter of fact the angle turned through by the co-tidal line is, if anything, rather greater than this, because the true co-tidal line for 8h. 10m. must be a little north of Bardsey and south of South Arklow, while the true co-tidal line for 6h. 15m. must run from a point slightly north of Tuskar Rock to a point slightly south of Ramsey Island. The true angle between the co-tidal lines for 6h. 15m. and 8h. 10m. is therefore slightly greater than $36\frac{1}{2}$°.

Now let us turn to the explanation. It has been pointed out already (see p. 9) that the "geostrophic" or deflecting force due to the earth's rotation increases the rise and fall of tide on the side of the channel which lies on the right-hand side of an observer who faces in the direction in which the tidal stream is running at H.W. In the case of a progressive tidal wave the current at high water is moving in the direction in which the wave is travelling.

The amplitude due to the in-going wave is therefore greater on the Welsh Coast than on the Irish Coast. In the case of the out-going wave, the right-hand side of the channel is the Irish Coast. The amplitude due to the out-going wave is therefore greater on the Irish Coast. The result of this is that the ratio of the amplitudes of the two waves is very much greater on the Welsh Coast than it is on the Irish side of the channel. The consequence is that the rate of travel of the co-tidal line near the point where the two waves oppose, *i.e.*, near the section AB, is much less on the Irish Coast than it is on the Welsh Coast.

This explanation can be verified quantitatively. Let y be the distance of any point from the central line of the South Channel, *i.e.*, from the line LM (fig. 3) joining the mid points of the two sections AB and TS. x and y are then co-ordinates of any point in the South Channel.

Since the tidal currents in the South Channel flow straight backwards and forwards without any appreciable circulatory motion, the increase in the height of the tidal oscillation on the right-hand side of the advancing wave can be expressed approximately in the form

$$h = a\left(1 - 2\,\frac{\omega y}{c}\,\sin\lambda\right)\cos\frac{2\pi}{\mathrm{T}}\left(t - \frac{x}{c}\right),^{*}$$

while the out-going wave is

$$b\left(1 + \frac{2\omega y\sin\lambda}{c}\right)\cos\frac{2\pi}{\mathrm{T}}\left(t + \frac{x}{c}\right).$$

The height of the tide at the point (x, y) and at time, t, is therefore given by

$$h = a\left(1 - \frac{2\omega y\sin\lambda}{c}\right)\cos\frac{2\pi}{\mathrm{T}}\left(t - \frac{x}{c}\right) - b\left(1 + \frac{2\omega y\sin\lambda}{c}\right)\cos\frac{2\pi}{\mathrm{T}}\left(t + \frac{x}{c}\right).$$

It is evident that all the analysis given above respecting the rate of travel of the co-tidal line still applies for any fixed value of y provided we use $a\left(1 - \frac{2\omega y\sin\lambda}{c}\right)$ instead of a, and $b\left(1 + \frac{2\omega y\sin\lambda}{c}\right)$ instead of b. The actual values of a and b which we found apply to the middle line, $y = 0$.

Denoting $a\left(1 - \frac{2\omega y\sin\lambda}{c}\right)$ by h_1, and $b\left(1 + \frac{2\omega y\sin\lambda}{c}\right)$ by h_2, where h_1 and h_2 are functions of y, the equation for the co-tidal line for 6h. 15m. is evidently found from (41) by replacing a and b by h_1 and h_2. The equation in question is

$$\cot\frac{2\pi t_x}{\mathrm{T}} = \frac{h_1 - h_2}{h_1 + h_2}\cot\frac{2\pi x}{c\mathrm{T}}. \qquad (46)$$

where t_x is constant but h_1, h_2 and x vary.

* This is evident from the analysis given on p. 10, but reference may also be given to LAMB'S 'Hydrodynamics,' p. 304, 1906 edition, where the expression $\zeta = ae^{-\frac{2\omega y}{c}}\cos \mathrm{K}\,(cb - x)$ occurs for the height of a long wave in a long rotating canal. The above expression is an obvious modification of this.

For this analysis to be correct, the co-tidal line at $x = 0$, *i.e.*, the line for 8h. 10m. should be perpendicular to the direction in which the wave has been assumed to be moving, *i.e.*, perpendicular to the middle line of the channel. As a matter of fact the angle between the co-tidal line AB, and the central line LM, differs considerably from a right angle. This is no doubt due partly to modifications introduced by the fact that the channel has not got parallel sides, but more probably it is due to the fact that the tidal wave from the Atlantic does not strike the channel in such a way as to allow the co-tidal line for 6h. 15m. to be at such an angle with the direction of the middle line of the channel as to allow it to become perpendicular to the channel (owing to the co-tidal line travelling faster on the Welsh side than on the Irish side) when it has travelled up the channel as far as the line AB.

It is worth while, however, to apply equation (46) to find out what angle the co-tidal line would have turned through, theoretically, in the time from 6h. 15m. to 8h. 10m.

The angle, θ, between the co-tidal line for 6h. 15m. and the co-tidal line for 8h. 10m. should be given by

$$\tan\theta = \frac{dx}{dy} \qquad (47)$$

where $\frac{dx}{dy}$ is obtained by differentiating (46).

Turning now to the figures, we have seen (see equation 44) that

$$\cot\frac{2\pi t_x}{\mathrm{T}} = -0{\cdot}67.$$

Hence (46) becomes

$$\cot\frac{2\pi x}{c\mathrm{T}} = -0{\cdot}67\,\frac{\frac{h_1}{h_2}+1}{\frac{h_1}{h_2}-1} \qquad (48)$$

But

$$\frac{h_1}{h_2} = \frac{a\left(1-\frac{2\omega y\sin\lambda}{c}\right)}{b\left(1+\frac{2\omega y\sin\lambda}{c}\right)} \qquad (49)$$

and if we limit ourselves to the consideration of the angle through which the co-tidal line turns during its passage up the central part of the channel, *i.e.*, up the line ML, y may be considered as small. In this case (49) may be written approximately

$$\frac{h_1}{h_2} = \frac{a}{b}\left(1-\frac{4\omega y\sin\lambda}{c}\right). \qquad (50)$$

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Differentiating (48) it will be found that

$$-\frac{2\pi}{c\mathrm{T}}\operatorname{cosec}^2\frac{2\pi x}{c\mathrm{T}}\frac{dx}{d\left(\frac{h_1}{h_2}\right)} = \frac{0{\cdot}67\times 2}{\left(\frac{h_1}{h_2}-1\right)^2} \quad \ldots \ldots \quad (51)$$

Since y is small, h_1 and h_2 may be taken as a and b in places where they are not being differentiated. Hence, since $a/b = 1{\cdot}8$*, the right-hand side of (51) is equal to

$$\frac{0{\cdot}67\times 2}{(1{\cdot}8-1)^2} = 2{\cdot}1, \text{ and (51) becomes } \frac{dx}{d\left(\frac{h_1}{h_2}\right)} = -\frac{2{\cdot}1c\mathrm{T}}{2\pi\operatorname{cosec}^2\frac{2\pi x}{c\mathrm{T}}} \quad \ldots \ldots \quad (52)$$

Differentiating (50)

$$\frac{d}{dy}\left(\frac{h_1}{h_2}\right) = -\frac{a}{b}\,\frac{4\omega\sin\lambda}{c} \quad \ldots \ldots \quad (53)$$

Combining (52) and (53) with (47)

$$\tan\theta = \frac{dx}{dy} = \frac{2{\cdot}1c\mathrm{T}}{2\pi\operatorname{cosec}^2\frac{2\pi x}{c\mathrm{T}}}\times\frac{a}{b}\,\frac{4\omega\sin\lambda}{c}) \quad \ldots \ldots \quad (54)$$

In (54), T = 12·4h., and from (43),

$$\cot\frac{2\pi x}{c\mathrm{T}} = -2{\cdot}44,$$

hence

$$\operatorname{cosec}^2\frac{2\pi x}{c\mathrm{T}} = 1 + (2{\cdot}44)_2 = 7,$$

also

$$\frac{a}{b} = 1{\cdot}8.$$

ω, the angular velocity of the earth in radians per hour is $\frac{2\pi}{24}$, and $\sin\lambda = 0{\cdot}79$.

Hence

$$\tan\theta = \frac{2{\cdot}1\times 12{\cdot}4\times 1{\cdot}8\times 4}{7\times 2\pi}\times\frac{2\pi}{24}\times 0{\cdot}79 = 0{\cdot}88$$

and

$$\theta = 41^\circ.$$

The agreement between this and the measured angle, $36\frac{1}{2}^\circ$, is quite as good as could be expected.

* See equation (45).

Effect of the Shape of the Coast in Determining the Time of H. W. at Points on the Coast.

This is too complicated a matter to treat quantitatively, there is, however, one point in connection with local peculiarities of the tides of the South Channel of the Irish Sea which can be explained qualitatively by the analysis contained in this paper, and that is the effect of a point of land projecting into the channel in altering the times of H.W. on the two sides of it.

Consider the effect of a point of land on the range of tide due to a tidal wave passing along a channel. Let AB, fig. (4), represent a portion of one side of a channel and let CDE be a point of land projecting into it.

Suppose the tidal wave moves along in the direction from A to B. At H.W. the tidal stream is moving from A to B, it might be expected therefore that, owing to the piling up of the water on side facing the current, the level of the water would be higher at C than at E at H.W., that is to say, the range of tide should be greater

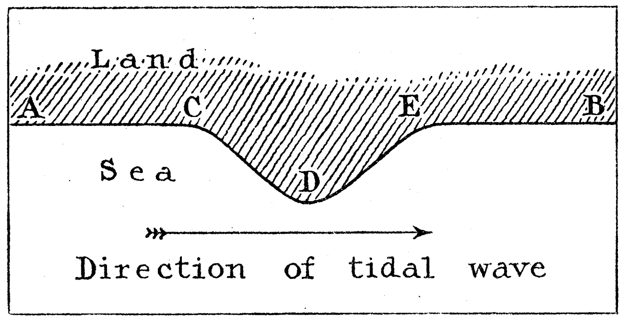

Fig. 4. Effect of a cape on times of H.W. on either side of it.

on the side of the cape which faces the direction from which the tidal wave comes than it would be on the side which faces away from the direction of the tidal wave. This effect does not materially alter the time of H.W. when there is only one tidal wave in the channel. When, however, there are two nearly equal waves, one going up and the other going down, the case is altered.

The time t_x of H.W. at distance x is given by

$$\cot \frac{2\pi t_x}{\mathrm{T}} = \frac{a-b}{a+b} \cot \frac{2\pi x}{c\mathrm{T}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Suppose now, that, without altering x, a is decreased while b is increased by the action of some local peculiarity* as it is on the side DE of the cape (fig. 4).

If x is positive and $\frac{2\pi x}{c\mathrm{T}} < \frac{\pi}{2}$ then equation (41) shows that a decrease in a and an increase in b will lead to an increase in t_x, the time of H.W.

* This increase must not be so great as to reverse the tides.

Similarly on the down-channel side, CD, of the cape, the time of H.W. will be made earlier by this local peculiarity.

When x is negative and when $-\frac{2\pi x}{c\text{T}} < \frac{\pi}{2}$ a decrease in a and an increase in b leads to an increase in $-t_x$, *i.e.*, to a decrease in the time of H.W. Similarly, on the down-channel face of the cape, H.W. is made late by the cape.

There are two interesting examples of this on the S.E. coast of Ireland. One is at Wicklow Head. This is situated in the region where x is positive. We should, therefore, expect it to be H.W. later on the northern side of the cape than on the southern side. It is H.W. at Wicklow, a few miles N. of the cape at 10·53, 2h. 30m. later than H.W. at Arklow, some 11 miles south of the cape. This effect evidently appears to make the co-tidal line travel very slowly past Wicklow Head.

The other example is that of Greenore Point. In this case x is negative, we should therefore expect the effect of the coast line to be to make the time of H.W. earlier on the north side of the cape than on the southern side.

This effect might, if it were sufficiently great, reverse the direction of travel of the co-tidal line in the neighbourhood of the point. As a matter of fact the effect is great enough to do this. It is H.W. at Saltees, some 10 miles S.W. of Carnsore Point, at 6h. 6m. At Carnsore Point, 4 miles S. of Greenore Point, it is H.W. at 6h. 25m. At Tuskar Rock, 4 miles out from Greenore Point, it is H.W. at 6h. 10m. At Wexford South Bay, on the north side of the Point, it is H.W. at 6h. 5m. After that the time of H.W. gets later as one goes further northwards up the coast. It will be seen that from Carnsore Point to Wexford South Bay, therefore, the direction of travel of the co-tidal line is just reversed. The fact that H.W. at Wexford South Bay, which is well round Greenore Point, is actually earlier than H.W. at Tuskar Rock which is south of Greenore Point, besides being 4 miles out at sea, is remarkable. I do not know whether any explanation has been offered before of how it is that the effect of a cape on the tidal phenomena in its neighbourhood is so very different in different parts of the sea.

Summary of Conclusions. (*Added October 24, 1919.*)

The rate of dissipation of energy at spring tides in the Irish Sea is calculated from the known formulæ for skin-friction of the wind on the ground and the friction of rivers on their beds. The results range from 1040 to 1300 ergs per square centimetre per second. The least of these is 150 times as great as Mr. STREET's previous estimate of 7 ergs per square centimetre per second.

The rate at which energy flows into the Irish Sea is next calculated from the rise and fall of tide, the strength of the tidal current and their phase difference over two sections taken across the North and South Channels. The rate of dissipation of energy is found to be 1530 ergs per square centimetre per second. This is in good agreement with the previous result.

It is next shown that this absorption of energy is sufficient to reduce the amplitude of the in-coming wave to one-half, so that three-quarters of the energy of the in-coming tidal wave is absorbed.

This absorption of energy explains most of the chief characteristics of the tidal phenomena of the South Channel to the Irish Sea, the velocity of the co-tidal line, which is only about one-third of the velocity of the tidal wave, the angle through which the co-tidal line turns in passing up the channel and the effect of Carnsore Point and Wicklow Head on the times of H.W. to the north and south of them.

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